Note to Class

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Two notes for the price of one today. (If you have a fancy enough PDF reader you can click on the red links to skip ahead to the relevant parts.

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1 Sums of k-th powers and integrals of polynomials

Today in class I tried presenting a nice way of figuring out sums of the form

$$\sum_{i=1}^{n} i^k.$$

I'd just come up with the idea to present things that way on the way to my 11:00 AM section, so I hadn't had a chance to sit down and work out the calculation, and I think I dropped a minus sign somewhere and gave up on that so we'd have time to go over the homework. I gave a much better presentation in my 12:00 PM section on the same material, so here, by way of apology, is a summary of what I said there.

1.1 Adding the numbers from 1 to n

Consider finding the sum $1 + 2 + \cdots + 10$. We could add these numbers up directly, but it would be nice if we could use some kind of formula to do this instead. Writing out the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10,$$

we can clearly make five pairs of numbers that add up to eleven:

$$(1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6).$$

Giving us a total of five elevens, or 55. If we want to add up the numbers from 1 to n instead, we can do the same thing. We'll have $n/2$ pairs that add up to $n + 1$, for a total of

$$1 + 2 + \cdots + n = \frac{n}{2}(n + 1) = \frac{1}{2}n^2 + \frac{1}{2}n.$$

For instance, if we wanted to add up the first million numbers, we'd get

$$1 + 2 + \cdots + 1,000,000 = 500,000,500,000.$$

Here's another way to see this, which will be easier to generalize. If we write $f(n) = \frac{1}{2}n^2 + \frac{1}{2}n$, then what we've just said is that

$$\begin{aligned} 1 + 2 + \cdots + (n-1) \quad +n \quad &= f(n), \qquad \text{and} \\ 1 + 2 + \cdots + (n-1) \qquad\quad &= f(n-1). \end{aligned}$$

In other words, if we subtract these, it says that

$$f(n) - f(n-1) = n.$$

1.2 Sums of squares and higher powers

The previous suggests suggest that, in order to produce a function $g(n)$ which is equal to the sum of the first n squares, we should try to find a function with

$$g(n) - g(n-1) = n^2.$$

Since $f(n)$ was a polynomial, it makes sense to suppose that $g(n)$ might also be a polynomial. Let's see where we can go with this. Instead of trying to find polynomials that satisfy this condition all at once, let's just take some simple polynomials and calculate $g(n) - g(n-1)$. In particular, let's just try the simplest polynomials, namely the ones of the form $g(x) = n^k$.

$g(n)$	$g(n) - g(n-1)$	
1	$1 - 1$	$= 0$
n	$n - (n-1)$	$= 1$
n^2	$n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1)$	$= 2n - 1$
n^3	$n^3 - (n-1)^3 = n^2 - (n^3 - 3n^2 + 3n - 1)$	$= 3n^2 - 3n + 1$

The calculation on the second line tells us the (admittedly obvious) fact that

$$\sum_{i=1}^{n} 1 = n,$$

while the one on the third says that

$$\sum_{i=1}^{n} 2i - 1 = n^2.$$

Expanding this a bit, we see that

$$2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = n^2.$$

But we know what the sum $1 + 1 + \cdots + 1$ is, so we can write this as

$$2\left(\sum_{i=1}^{n} i\right) - n = n^2.$$

Solving, we have

$$\sum_{i=1}^{n} i = \frac{n^2 + n}{2}.$$

Of course this is just the same formula we had above. But now we can do the same trick for the sums of cubes. The fourth line of the table above tells us that

$$\sum_{i=1}^{n} 3n^2 - 3n + 1 = n^3.$$

Expanding,

$$3\sum_{i=1}^{n} n^2 - 3\sum_{i=1}^{n} n + \sum_{i=1}^{n} 1 = n^3.$$

Substituting in the parts we already know,

$$3\left(\sum_{i=1}^{n} n^2\right) - 3\left(\frac{n^2+n}{2}\right) + n = n^3.$$

Solving,

$$\sum_{i=1}^{n} n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

(It probably looks like a lot more work that it really was – nearly all of those equations are just copying down the line above and plugging something in or moving something to the other side.)

1.3 Computing an integral

Okay, let's see how to use this to compute an integral. In particular, let's look at $\int_0^4 x^2\,dx$. By definition, we just take Riemann sums – let's say right Riemann sums:

$$\begin{aligned}
\int_0^4 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{4}{n}i\right)\frac{4}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^2 \frac{4}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i^2}{n^2}\frac{4}{n} \\
&= \lim_{n\to\infty} \frac{64}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{64}{n^3}\left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right] \\
&= \lim_{n\to\infty} \frac{64}{3} + \frac{32}{n} + \frac{32}{3n^2} = \frac{64}{3}
\end{aligned}$$

Hopefully this helps illustrate what Riemann sums are actually "for." We got an actual formula for the Riemann sum, and then we were able to take the limit and compute the actual integral. While we're here, let's go ahead and work the same problem using the Fundamental Theorem of Calculus:

$$\int_0^4 x^2\,dx = \left.\frac{1}{3}x^3\right|_0^4 = \frac{4^3}{3} - \frac{0^3}{3} = \frac{64}{3}.$$

2 Another perspective on a l'Hôpital's rule problem

In class today, I demonstrated how to compute the limit

$$\lim_{x\to 0^+} x^2 \ln(x)$$

by rewriting it as

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x^2}}.$$

Someone suggested that I might instead make the substitution $y = e^x$, rendering the limit as

$$\lim_{y \to -\infty} y e^{2y}.$$

We can use a similar trick here

$$\lim_{y \to -\infty} \frac{y}{e^{-2y}}.$$

The numerator and denominator each approach infinity, so we can apply l'Hôpital's rule, giving

$$\lim_{y \to -\infty} \frac{1}{-2e^{-2y}} = 0.$$

l'Hôpital's rule, by the way, is now believed to have been developed by Johann Bernoulli. l'Hôpital had for some unknown reason offered Bernoulli ridiculous amounts of money in exchange for keeping his mathematical discoveries secret and only communicating them to l'Hôpital – bizarrely, since there was no chance of, say, making much money off of them. Apparently l'Hôpital's goal was to take implicit credit for discovering the ideas when he put them into the textbooks he wrote. And he got away with it for a few hundred years, even after Bernoulli went public after l'Hôpital's death. Then Bernoulli's original papers were discovered in the 1920's, shedding a good deal more light on the situation.